SOME CONTACT PROBLEMS FOR THE ELASTIC LAYER

(NEKOTORYE KONTAKTNYE ZADACHI DLIA UPRUGOGO SLOIA)

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At present one may consider as experimentally established that the best approximation to the reality is given by considering the foundation as an elastic layer of finite thickness. Therefore the solution of contact problems for the elastic layer acquires a great interest.

In the present paper problems are considered on the penetration of a rigid stamp into the elastic layer lying without friction upon an undeformable base or rigidly connected with the undeformable base. It is assumed that (1) the region Ω of contact of the stamp with the layer is arbitrary and fixed, (2) the friction forces between the stamp and the layer are absent, (3) outside the stamp the layer is not loaded. It is shown that an approximate solution of the indicated contact problem for any layer of thickness h can be found if the solution of the corresponding contact problem for the elastic semi-space is known. Tables for obtaining approximate solutions are given. As an example, the case of an elliptic contact region is considered.

1. Integral equations, properties of their kernels. By means of the methods of the variational calculus the above indicated problems can be reduced to the determination of the contact pressure $q(P)$ between the stamp and the layer from the integral equation

$$\int_{\Omega} q(P)\, K(R / h)\, dP = 2\pi \Delta h g(Q), \qquad Q \in \Omega \qquad \left(\Delta = \frac{E}{2(1-\sigma^2)}\right) \tag{1.1}$$

Here R is the distance between the points P and Q; the function $g(Q)$ is the given setting of the layer boundary under the stamp, which is determined by the shape of the stamp base and by the degree of penetration of the stamp into the layer. Obviously, on physical grounds $g(Q) > 0$ for $Q \in \Omega$. Also, for the same reason, the conditions

$$q(P) > 0 \quad \text{when } P \in \Omega, \qquad \int_\Omega q(P)\,dP < \infty \tag{1.2}$$

must be satisfied.

The class of functions $g(Q)$, positive in Ω, for which there exists the solution of equation (1.1) satisfying conditions (1.2), we shall denote $G(h)$; and in what follows we shall assume that $g(Q) \in G(h)$.

We remark also that the pressure $q(P)$ is connected with the force and the moments acting upon the stamp by the known relations

$$N = \int_\Omega q(P)\,dP, \qquad M_\xi = \int_\Omega \eta q(P)\,dP, \quad M_\eta = \int_\Omega \xi q(P)\,dP \tag{1.3}$$

The kernel of equation (1.1) has the form for the case of the layer:

a) lying on the undeformable base without friction

$$K(R/h) = \int_0^\infty \frac{\cosh 2u - 1}{\sinh 2u + 2u} J_0(uR/h)\,du \tag{1.4}$$

b) rigidly connected with the undeformable base

$$K(R/h) = \int_0^\infty \frac{2(3-4\sigma)\sinh 2u - 4u}{2(3-4\sigma)\cosh 2u + (3-4\sigma)^2 + 1 + 4u^2} J_0(uR/h)\,du \tag{1.5}$$

Here $J_0(x)$ is the Bessel function of zero order.

It can be shown that the kernels (1.4) and (1.5) may be represented in the form

$$K(k) = k^{-1} - F(k) \qquad (-\infty < k = R/h < \infty) \tag{1.6}$$

In formula (1.6) the function $F(k)$ is even, continuous, and continuously differentiable any number of times; it may be represented as a power series with the convergence radius $\rho = 2$

$$F(k) = \sum_{i=0}^\infty a_i k^{2i}, \qquad |k| < 2 \tag{1.7}$$

It can be shown that for $|k| \to \infty$ of the kernel

$$K(k) \to 2\pi A \delta(k) \tag{1.8}$$

$$\left(A = \frac{1-2\sigma}{2(1-\sigma)^2}\right)$$

Here $\delta(k)$ is a two-dimensional delta function; A is given for the case (1.5); for the case (1.4), $A = 1/2$.

TABLE 1.

	σ	a_0	a_1	a_2	a_3	a_4
a		1.168	−0.395	0.129	−0.0379	0.0106
b	0.1	1.242	−0.503	0.200	−0.0711	0.0236
	0.2	1.292	−0.552	0.226	−0.0818	0.0272
	0.3	1.377	−0.627	0.265	−0.0976	0.0329
	0.4	1.519	−0.747	0.327	−0.123	0.0418

The values of the constants a_i are given in Table 1 with three significant figures. In Tables 2 and 3 are given the values of the function $F(k)$ computed on the computer "Ural" corresponding to the problems (1.4) and (1.5). The tables are computed so that the intermediate values of $F(k)$ can be obtained by linear interpolation with three exact figures after the decimal point.

2. Degenerate solutions for very large and very small h, solution for large h. Using formula (1.6) let us represent (1.1) in the form

$$\int_{\Omega} \frac{q(P)}{R} dP = 2\pi\Delta g(Q) + \frac{1}{h}\int_{\Omega} q(P) F(R/h)\, dP \qquad (Q \in \Omega) \tag{2.1}$$

For very large values of h one may neglect the second term on the right-hand side of the equation (2.1) which then becomes

$$\int_{\Omega} \frac{q(P)}{R} dP = 2\pi\Delta g(Q) \qquad (Q \in \Omega) \tag{2.2}$$

As is known, the problem of the action of the stamp upon an elastic semi-space ($h = \infty$) is reduced to equation (2.2), however the solution of equation (2.2) may be considered as an approximate one for very large $h < \infty$. We remark also that the solution of equation (1.1) or (1.2) for any value $h \in (0, \infty)$ will obviously be of the same character (will have the same properties) as the solution of equation (2.2). This follows from the fact that the right-hand side of equation (2.1) differs from the right-hand side of equation (2.2), due to the properties of the function $F(k)$ and conditions (1.2), by a function which is continuous and continuously differentiable any number of times. For the case when the region Ω is an ellipse and the function $g(Q)$ is an arbitrary polynomial, the solution of equation (2.2), bounded and unbounded on the contour L

of the region Ω, is obtained in the author's [1] paper*.

TABLE 2.

k	$F(k)$	k	$F(k)$	k	$F(k)$	k	$F(k)$	k	$F(k)$
0.00	1.168	0.75	0.980	1.50	0.666	2.25	0.453	3.00	0.335
0.05	1.167	0.80	0.959	1.55	0.648	2.30	0.443	3.05	0.330
0.10	1.164	0.85	0.937	1.60	0.630	2.35	0.433	3.10	0.324
0.15	1.159	0.90	0.915	1.65	0.613	2.40	0.423	3.15	0.319
0.20	1.152	0.95	0.893	1.70	0.597	2.45	0.414	3.20	0.314
0.25	1.143	1.00	0.871	1.75	0.581	2.50	0.406	3.25	0.309
0.30	1.133	1.05	0.849	1.80	0.566	2.55	0.397	3.30	0.304
0.35	1.121	1.10	0.827	1.85	0.551	2.60	0.389	3.35	0.299
0.40	1.107	1.15	0.806	1.90	0.537	2.65	0.382	3.40	0.295
0.45	1.092	1.20	0.784	1.95	0.524	2.70	0.374	3.45	0.290
0.50	1.076	1.25	0.764	2.00	0.511	2.75	0.367	3.50	0.286
0.55	1.059	1.30	0.743	2.05	0.498	2.80	0.360		then as $1/k$
0.60	1.040	1.35	0.723	2.10	0.486	2.85	0.354		
0.65	1.021	1.40	0.703	2.15	0.474	2.90	0.347		
0.70	1.001	1.45	0.684	2.20	0.463	2.95	0.341		

For very small values of h equation (1.1), because of (1.8), may be represented in the form

$$A \int_{\Omega} q(P)\, \delta(R/h)\, dP = \Delta h g(Q) \qquad (Q \in \Omega) \tag{2.3}$$

The solution of equation (2.3) is found without difficulty

$$q(Q) = \Delta g(Q) / Ah \qquad (Q \in \Omega) \tag{2.4}$$

It is advisable to use the solution of equation (2.2) as an approximate solution of equation (1.1) for very large h when $h \geqslant \chi_\infty a$, and the degenerate (limit) solution (2.4) for very small h, when $h \leqslant \chi_0 a$ ($a = 1/2 \max_\Omega R$). The values χ_∞ and χ_0 are given in Table 4 for the problems (a) and (b). Thereby the values χ_0 are computed for stamps which in plane are close to circular. Within the limits indicated the obtained approximate solutions may be considered as practically exact.

For large values of h, namely for $h > a$, the magnitude $|k| < 2$ and, consequently, the function $F(k)$ may be represented in the form (1.7). Keeping this in mind let us rewrite equation (2.1)

$$\int_{\Omega} q(P) \frac{dP}{R} = 2\pi \Delta g(Q) + \sum_{i=0}^{\infty} \frac{a_i}{h^{2i+1}} \int_{\Omega} q(P) R^{2i} dP \qquad (Q \in \Omega) \tag{2.5}$$

* Some particular solutions for this case have been obtained previously in the papers [2-4].

TABLE 3.

k	$F(k)$	k	$F(k)$	k	$F(k)$	k	$F(k)$	k	$F(k)$
$\sigma = 0.1$									
0.00	1.242	0.70	1.036	1.40	0.701	2.10	0.480	2.80	0.358
0.05	1.240	0.75	1.011	1.45	0.681	2.15	0.469	2.85	0.352
0.10	1.236	0.80	0.986	1.50	0.661	2.20	0.458	2.90	0.346
0.15	1.230	0.85	0.961	1.55	0.642	2.25	0.448	2.95	0.340
0.20	1.222	0.90	0.935	1.60	0.624	2.30	0.438	3.00	0.334
0.25	1.211	0.95	0.910	1.65	0.607	2.35	0.429	3.05	0.329
0.30	1.198	1.00	0.885	1.70	0.591	2.40	0.420	3.10	0.323
0.35	1.183	1.05	0.860	1.75	0.575	2.45	0.411	3.15	0.318
0.40	1.166	1.10	0.836	1.80	0.559	2.50	0.402	3.20	0.313
0.45	1.147	1.15	0.812	1.85	0.545	2.55	0.394	3.25	0.308
0.50	1.127	1.20	0.788	1.90	0.531	2.60	0.387		then as
0.55	1.106	1.25	0.765	1.95	0.517	2.65	0.379		$1/k$
0.60	1.083	1.30	0.743	2.00	0.504	2.70	0.372		
0.65	1.060	1.35	0.722	2.05	0.492	2.75	0.365		
$\sigma = 0.2$									
0.00	1.292	0.65	1.094	1.30	0.754	1.95	0.519	2.60	0.386
0.05	1.291	0.70	1.068	1.35	0.731	2.00	0.506	2.65	0.379
0.10	1.287	0.75	1.041	1.40	0.709	2.05	0.493	2.70	0.372
0.15	1.280	0.80	1.014	1.45	0.688	2.10	0.481	2.75	0.365
0.20	1.271	0.85	0.986	1.50	0.668	2.15	0.470	2.80	0.358
0.25	1.259	0.90	0.959	1.55	0.648	2.20	0.459	2.85	0.352
0.30	1.245	0.95	0.932	1.60	0.629	2.25	0.448	2.90	0.346
0.35	1.228	1.00	0.905	1.65	0.611	2.30	0.438	2.95	0.340
0.40	1.210	1.05	0.878	1.70	0.594	2.35	0.428	3.00	0.334
0.45	1.189	1.10	0.852	1.75	0.578	2.40	0.419		then as
0.50	1.167	1.15	0.827	1.80	0.562	2.45	0.410		$1/k$
0.55	1.144	1.20	0.802	1.85	0.547	2.50	0.402		
0.60	1.120	1.25	0.777	1.90	0.532	2.55	0.394		
$\sigma = 0.3$									
0.00	1.377	0.80	1.063	1.60	0.641	2.40	0.422	3.20	0.314
0.05	1.375	0.85	1.032	1.65	0.622	2.45	0.413	3.25	0.309
0.10	1.370	0.90	1.002	1.70	0.604	2.50	0.404	3.30	0.304
0.15	1.363	0.95	0.972	1.75	0.586	2.55	0.396	3.35	0.299
0.20	1.352	1.00	0.942	1.80	0.570	2.60	0.388	3.40	0.295
0.25	1.339	1.05	0.912	1.85	0.554	2.65	0.380	3.45	0.291
0.30	1.322	1.10	0.883	1.90	0.539	2.70	0.373	3.50	0.286
0.35	1.304	1.15	0.855	1.95	0.524	2.75	0.366	3.55	0.282
0.40	1.283	1.20	0.828	2.00	0.511	2.80	0.359	3.60	0.278
0.45	1.260	1.25	0.802	2.05	0.498	2.85	0.353	3.65	0.274
0.50	1.235	1.30	0.776	2.10	0.485	2.90	0.347	3.70	0.271
0.55	1.209	1.35	0.751	2.15	0.473	2.95	0.341	3.75	0.267
0.60	1.181	1.40	0.727	2.20	0.462	3.00	0.335		then as
0.65	1.153	1.45	0.704	2.25	0.451	3.05	0.329		$1/k$
0.70	1.123	1.50	0.682	2.30	0.441	3.10	0.324		
0.75	1.093	1.55	0.662	2.35	0.431	3.15	0.319		

$\sigma = 0.4$

k	$F(k)$	k	$F(k)$	k	$F(k)$	k	$F(k)$	k	$F(k)$
0.00	1.519	1.15	0.908	2.30	0.450	3.45	0.293	4.60	0.218
0.05	1.517	1.20	0.877	2.35	0.439	3.50	0.289	4.65	0.216
0.10	1.511	1.25	0.846	2.40	0.429	3.55	0.285	4.70	0.214
0.15	1.502	1.30	0.817	2.45	0.420	3.60	0.281	4.75	0.212
0.20	1.489	1.35	0.789	2.50	0.411	3.65	0.277	4.80	0.209
0.25	1.473	1.40	0.762	2.55	0.402	3.70	0.273	4.85	0.207
0.30	1.454	1.45	0.737	2.60	0.394	3.75	0.269	4.90	0.205
0.35	1.432	1.50	0.712	2.65	0.386	3.80	0.266	4.95	0.203
0.40	1.407	1.55	0.689	2.70	0.378	3.85	0.262	5.00	0.201
0.45	1.380	1.60	0.667	2.75	0.371	3.90	0.259	5.05	0.199
0.50	1.350	1.65	0.645	2.80	0.364	3.95	0.255	5.10	0.197
0.55	1.319	1.70	0.625	2.85	0.358	4.00	0.252	5.15	0.195
$\sigma = 0.4$									
0.60	1.287	1.75	0.606	2.90	0.351	4.05	0.249	5.20	0.193
0.65	1.253	1.80	0.588	2.95	0.345	4.10	0.246	5.25	0.191
0.70	1.219	1.85	0.571	3.00	0.339	4.15	0.243	5.30	0.189
0.75	1.183	1.90	0.554	3.05	0.333	4.20	0.240	5.35	0.188
0.80	1.148	1.95	0.539	3.10	0.328	4.25	0.237	5.40	0.186
0.85	1.112	2.00	0.524	3.15	0.322	4.30	0.234	5.45	0.184
0.90	1.077	2.05	0.510	3.20	0.317	4.35	0.231	5.50	0.182
0.95	1.042	2.10	0.497	3.25	0.312	4.40	0.229		then as
1.00	1.007	2.15	0.484	3.30	0.307	4.45	0.226		$1/k$
1.05	0.973	2.20	0.472	3.35	0.302	4.50	0.224		
1.10	0.940	2.25	0.460	3.40	0.298	4.55	0.221		

TABLE 4.

	(a)	(b)			
σ		0.1	0.2	0.3	0.4
χ_∞	15.5	16.5	17	18.5	20.5
χ_0	$\frac{1}{8}$	$\frac{1}{7.5}$	$\frac{1}{7}$	$\frac{1}{8.5}$	$\frac{1}{12.5}$

The approximate solution of equation (2.5) may be found either by the method of successive approximations or by the method presented in [5], if the solution of equation (2.2) is known. It is advisable to use the approximate solutions thus obtained for $h \geqslant 1.5 \div 2a$. As an example, [5] contains an approximate solution of equation (2.5) for the case of an elliptic region Ω and

$$g(Q) = \gamma + \alpha x + \beta y$$

We remark that from the practical viewpoint it is often sufficient to determine not the solution of equation (1.1) itself, but the magnitude of the force N acting upon the stamp, or at least an estimate of this magnitude.

Let us show that one may obtain a two-sided estimate of the magnitude of the force N, if one knows the solution $q(P) = \nu q_\nu(P)$ of equation (2.2) for $g(Q) = \nu$, i.e. if the solution of the problem of the action of a flat stamp upon the elastic semi-space is known.

Multiply both sides of equation (2.1) by $\nu q_\nu(Q)$ and integrate with respect to Ω. After this, transposing the integrals on the left-hand side, following [6], we obtain

$$N = J(g) + \frac{1}{2\pi\Delta h} J\left(\int_\Omega q(P) F(R/h)\, dP\right) \qquad \left(J(f) = \int_\Omega q_\nu(Q) f(Q)\, dQ\right) \quad (2.6)$$

Now from (2.6) we find without difficulty the two-sided estimate of the magnitude of the force

$$\frac{J(g)}{1 - (J(1)/2\pi\Delta h)\min_\Omega F(k)} \leqslant N \leqslant \frac{J(g)}{1 - (J(1)/2\pi\Delta h)\max_\Omega F(k)} \qquad (2.7)$$

The two-sided estimate obtained is, generally speaking, correct for any value of h, but it is sufficiently effective only for large h, about $h \geqslant 1.5 \div 2a$. For a given h the values $\max_\Omega F(k)$ and $\min_\Omega F(k)$ can easily be found from Tables 2 and 3.

Let the region Ω be an ellipse with semi-axes a and b, and $g(Q) = \gamma$, then formula (2.7) becomes

$$\frac{2\pi a\Delta\gamma}{K(e) - (a/h)\min_\Omega F} \leqslant N \leqslant \frac{2\pi a\Delta\gamma}{K(e) - (a/h)\max_\Omega F} \qquad (2.8)$$

where $K(e)$ is the complete elliptic integral of the first kind, e is the excentricity. Obviously, the mean value of the force N may be represented in the form

$$N_* = 4a\varkappa\Delta\gamma$$

Some results of computations are given in Table 5.

TABLE 5.

e^2	h/a	$\varkappa$	h/a	$\varkappa$
Problem (a)				
0.00	2	1.49(1.51)	4	1.22(1.22)
0.70	2	1.01(1.03)	4	0.88(0.88)
0.99	2	0.49(0.50)	4	0.46(0.46)
Problem (b) for $\sigma = 0.3$				
0.00	2	1.60(1.63)	4	1.26(1.27)

In parenthesis are given, for comparison, practically exact values of the magnitude κ, calculated according to the corresponding formula of [5].

We remark that a more exact estimate than (2.7), and a sufficiently convenient two-sided estimate for the force magnitude N and two-sided estimates for the moments M_ξ and M_η, are obtained in [7] at $h \geqslant 1.5 \div 2a$, for the special case when the region Ω has two mutually perpendicular axes of symmetry, and the function $g(Q)$ has the form $g(Q) = \gamma + \alpha x + \beta y - \varphi(x^2, y^2)$. Besides, it is assumed that the solution of equation (2.2) for $g(Q) = \nu + \lambda x + \mu y$ is known. As an example, [7] contains the two-sided estimate of the force magnitude for a flat stamp circular in the plane.

3. Approximate solution for intermediate values of h. For intermediate values of h ($1/7 \div 1/12.5a \leqslant h \leqslant 1.5 \div 2a$) the approximate solution of equation (1.1) or (2.1) can be found in the following way. Let us fix h, then $0 \leqslant |k| \leqslant 2a/h$, and let us in some manner approximate the function $F(k)$ for $|k| \in [0, 2a/h]$ by a polynomial (e.g. interpolation polynomial)

$$F(k) = \sum_{i=0}^{n} b_i k^{2i} \tag{3.1}$$

Substituting $F(k)$ from (3.1) into equation (2.1), let us represent it in the form

$$\int_\Omega q(P)\frac{dP}{R} = 2\pi\Delta g(Q) + \sum_{i=0}^{n} \frac{b_i}{h^{2i+1}} \int_\Omega q(P) R^{2i} dP \qquad (Q \in \Omega) \tag{3.2}$$

We show that, again, if the solution of equation (2.2) is known, then the solution of equation (3.2) is reduced to the solution of a system of linear algebraic equations. We seek the solution of equation (3.2) in the form

$$q(P) = q_0(P) + q_1(P) \tag{3.3}$$

where $q_0(P)$ is the solution of equation (2.2). It is easy to see then, that the correcting function $q_1(P)$ satisfies the equation

$$\int_\Omega q_1(P)\frac{dP}{R} = \sum_{i=0}^{n} \frac{b_i}{h^{2i+1}} \int_\Omega [q_0(P) + q_1(P)] R^{2i} dP \qquad (Q \in \Omega) \tag{3.4}$$

Let now $q_*(P)$ be the solution of equation (2.2) for

$$g(Q) = \sum_{i=0}^{i+j} \sum_{j=0}^{\leqslant 2n} c_{ij} x^i y^i \tag{3.5}$$

which, by assumption, is known. Setting $q_1(P) = q_*(P)$ and substituting into (3.4), we obtain, as is easy to see, on the left- and right-hand sides, polynomials of degree $2n$. Equating the coefficients of equal powers of x and y, we obtain a system of $(2n+1)(n+1)$ linear algebraic equations for determining $(2n+1)(n+1)$ unknowns c_{ij}. Solving the system and substituting the values found of c_{ij} into the expression $q_1(P)$, we obtain the approximate solution of equation (1.1) by the formula (3.3).

We remark that the thus obtained approximate solutions of equation (1.1) retain all the properties of the exact solution, since in replacing the function $F(k)$ by a polynomial, the properties of the kernel of this integral equation do not change.

As an example let us consider the case of an elliptic region of contact Ω with semi-axes a and b. Put $h = a$, $n = 1$ and $g(Q) = \gamma + \alpha x + \beta y$, then, obviously, the right-hand side of equation (3.2) is a polynomial of second degree, and, consequently, the solution of equation (3.2) for the case under consideration is to be sought in the form [3]

$$q(P) = \frac{\Delta}{b}\left(1 - \frac{\xi^2}{a^2} - \frac{\eta^2}{b^2}\right)^{-1/2}\left[\gamma\left(a_{00} + a_{20}\frac{\xi^2}{a^2} + a_{02}\frac{\eta^2}{b^2}\right) + \alpha a_{10}\xi + \beta a_{01}\frac{a^2}{b^2}\eta\right] \tag{3.6}$$

Thereby N, M_ξ and M_η will have the form

$$N = 4a\varkappa\Delta\gamma, \qquad \varkappa = {}^1/_2\,\pi\,[a_{00} + {}^1/_3\,(a_{20} + a_{02})] \tag{3.7}$$

$$M_\xi = {}^8/_3\,a^3\beta\,\Delta\psi_{01}, \quad \psi_{01} = {}^1/_4\,\pi a_{01}, \quad M_\eta = {}^8/_3\,a^3\alpha\,\Delta\psi_{10}, \quad \psi_{10} = {}^1/_4\pi a_{10} \tag{3.8}$$

Substitute $q(P)$ in the form (3.6) into equation (3.2) and evaluate the left-hand integral, using [1]; the right-hand integral is evaluated without difficulty. Equating then the coefficients of equal powers of x and y in relation (3.2), for determining the numbers a_{ij}, we obtain five linear algebraic equations

$$\begin{gathered}
a_{00}S_{00} + {}^1/_2\,a_{20}S_{01}(1 - e^2) + {}^1/_2 a_{02}S_{10} = 1 + a_{00}\,[b_0 + {}^1/_3 b_1(2 - e^2)] + \\
+ {}^1/_3 a_{20}\,[b_0 + {}^1/_5 b_1(4 - e^2)] + {}^1/_3 a_{02}\,[b_0 + {}^1/_5 b_1(4 - 3e^2)] \\
a_{20}S_{20} - {}^1/_2 a_{20}S_{11}(1 - e^2) + a_{02}S_{11}(1 - e^2) - {}^1/_2 a_{02}S_{20} = a_{00}b_1 + {}^1/_3 b_1 a_{20} + {}^1/_3 b_1 a_{02} \\
a_{20}S_{11} - {}^1/_2 a_{20}S_{02}(1 - e^2) + a_{02}S_{02}(1 - e^2) - {}^1/_2 a_{02}S_{11} = a_{00}b_1 + {}^1/_3 b_1 a_{20} + {}^1/_3 b_1 a_{02} \\
a_{10}S_{10} = 1 - {}^2/_3 a_{10}b_1, \qquad a_{01}S_{01} = 1 - {}^2/_3 a_{01}b_1
\end{gathered} \tag{3.9}$$

where e is the eccentricity and the S_{mn} have the form

$$S_{mn} = \int_0^{\pi/2} \frac{\cos^{2m}\varphi\,\sin^{2n}\varphi\,d\varphi}{(1 - e^2\sin^2\varphi)^{m+n+1/2}} \tag{3.10}$$

and can be expressed as complete elliptic integrals of the first and second kind.

Let us take as interpolation points on the segment $[0, 2]$ $k_0 = 0$ and $k_1 = 1.5$, then, using Tables 2 and 3, we find for the problem (a) $b_0 = 1.168$ and $b_1 = -0.223$, and for the problem (b), for $\sigma = 0,3$, $b_0 = 1.377$ and $b_1 = -0.309$. Now substitute the values b_0 and b_1 found into system (3.9) and solve it for $e^2 = 0.00$, 0.70 and 0.99 for the problem (a), and for $e^2 = 0.00$ for the problem (b).

The results of the solution, as well as the corresponding values of κ, ψ_{10} and ψ_{01}, are given in Table 6. For comparison, some values obtained in [8] are given in parenthesis.

TABLE 6.

	e^2	a_{00}	a_{10}	a_{01}	a_{20}	a_{02}	$\varkappa$	ψ_{10}	ψ_{01}
	0.00	2.001	1.571	1.571	−0.825	−0.825	2.28 (2.20)	1.23 (1.19)	1.23 (1.19)
a	0.70	1.056	0.959	0.357	−0.286	−0.131	1.44	0.75	0.28
	0.99	0.387	0.391	0.010	−0.038	−0.002	0.59	0.31	0.01
b	0.00	2.610	1.726	1.726	−1.347	−1.347	2.69 (2.48)	1.36 (1.27)	1.36 (1.27)

It is quite evident that the method of obtaining the approximate solution of equation (1.1) given in this section is applicable, generally speaking, for any values $h \in (0, \infty)$. However, when $h \geqslant 1.5 \div 2a$, it is more convenient to use the methods shown in Section 2 for very large and large values of h; when $h \leqslant 1/7 \div 1/12.5a$, then it is more rational to use the degenerate solution (2.4) for very small h, since to obtain approximate solutions of equal accuracy by the method of the present Section, it is necessary with decreasing h to increase n in equation (3.2) which leads to a rapid increase of the number of equations in the system.

Thus it is expedient to use each of the presented methods for the approximate solutions of the problems (a) and (b) on a certain definite interval of change of h, whereby all together they exhaust the entire range of change of $h \in [0, \infty]$, and may serve as a convenient means for engineering calculations.

In conclusion, we remark that the pressure $q(P)$ in the case of problem (b) turns out always to larger than in the problem (a) for equal setting of the stamp $g(Q)$, and all other conditions being equal; however this difference for $0 < \sigma \leqslant 0.2$ does not exceed 10 per cent.

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